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MINIMAL PAIRS IN INITIAL SEGMENTS OF THE RECURSIVELY ENUMERABLE DEGREES*

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ABSTRACT

We show that for every r.e. Turing degree $\mathbf{a} > \mathbf{0}$, there is an r.e. degree $\mathbf{b} < \mathbf{a}$ which is not half of a minimal pair in the initial segment $[\mathbf{0}, \mathbf{a}]$.

1. Introduction

Let $\mathcal{R}$ denote the uppersemilattice of r.e. degrees with operations $\cup$ (join) and $\cap$ (partial meet). A pair of r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ form a **minimal pair** in $\mathcal{R}$ if $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$ where $\mathbf{0}$ denotes the degree of the recursive sets. Minimal pairs were first constructed by Lachlan [Lac66] and by Yates [Yat65] independently.

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The construction of minimal pairs was important as it provided the first example of a construction of r.e. degrees with the meet operation controlled. As a result, it provided an important step in determining which lattices are embeddable in $\mathcal{R}$, a problem which still hasn't been completely solved.

This paper addresses the question of which recursively enumerable degrees bound minimal pairs. The first result along these lines is that of Cooper [Coo74] who showed that every high r.e. degree bounds a minimal pair. In a paper important not only for its results but also for the proof techniques used ("gap-cogap" and tree of strategy constructions), Lachlan showed that there is a nonzero r.e. degree which doesn't bound a minimal pair but also that there is a nonzero r.e. degree such that every nonzero r.e. degree below it bounds a minimal pair. This latter result was improved by Downey and Welch [DW86] and Ambos-Spies [AS84] who showed independently that there is a nonzero r.e. degree such that every r.e. degree below it is the join of a minimal pair. We extend these results in this paper by proving the following theorem. (NOTE: In the Theorem and throughout the paper, all sets and degrees are r.e.)

THEOREM 1.1: $(\forall \mathbf{a} \neq \mathbf{0})(\exists \mathbf{b} < \mathbf{a})(\forall \mathbf{c} \leq \mathbf{a})[\mathbf{b} \cap \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{c} = \mathbf{0}]$.

The general program of which this question is part is that of classifying the isomorphism types of initial segments of $\mathcal{R}$. While a complete classification is beyond present technology and is perhaps too much to expect, there is some hope of understanding natural fragments of the theories of such intervals. We see this paper as providing information of some importance in this direction. We already know that for instance the problem of embedding finite lattices is significantly different for initial segments and all of $\mathcal{R}$.

The decomposition of $\mathcal{R}$ into the definable ideal of the cappable degrees and filter of cuppable degrees by Ambos-Spies, Jockusch, Shore and Soare [AJSS84] plays a central role in trying to decide the existential-universal theory of Th($\mathcal{R}$). Therefore it is reasonable to expect that if one restricts oneself to intervals or initial segments of $\mathcal{R}$, then one needs to understand the role of the degrees cappable within the interval or initial segment. Natural questions arise upon taking this point of view. Are there cappable degrees relative to each interval/initial segment? (The answer is no by Lachlan [Lac79].) Is every degree in an interval either cappable or cuppable? (Again the answer is no by Downey [Dow87].) Do the cappable degrees in an interval form an ideal in that interval? (This question is open.) Is it possible that all degrees in an interval are cappable relative to that interval? The present paper contributes to the program by answering the last question negatively. We remark that the results so far seem to indicate that understanding the local behaviour of intervals/initial segments of $\mathcal{R}$, even at the most basic level of the Σ_2^0 theory, may be a significantly more difficult task than understanding that of $\mathcal{R}$.

In the next section, Section 2, we list the requirements necessary to prove Theorem 1.1 and describe the construction necessary to meet one requirement. In Section 3, we motivate and construct the priority tree for the full construction meeting all the requirements. In Section 4 we give the full construction. The verification that the construction works occupies Section 5.

We remark that the priority argument of this paper is of considerable technical interest. The construction is a tree of strategies argument of the kind that is now quite standard in the study of $\mathcal{R}$. It has one feature not found in previous constructions however. In this construction, as in other tree strategy constructions, we identify a true path along which the requirements of the construction are satisfied. However, it will be the case that there are nodes σ along this path such that σ is not visited in the construction infinitely often and so the requirement that σ is attempting to meet is not met at σ. The key point is that *somewhere* on the "genuine true path" (that portion which is visited infinitely often) there is a version of the requirement σ is attempting to meet. The existence of portions of the true path that might be visited only finitely often creates a number of interesting technical difficulties, since these portions might be encoding "incorrect information" about the behavior of requirements to the lower priority nodes. Thus the lower priority nodes not only need to know what the predecessors of the nodes are, but they also need to know the manner by which they were accessed.

Our terminology is quite standard; a reference is Soare [Soa87].

2. The requirements and basic module

To prove Theorem 1.1, suppose that an r.e. set A is given. We construct an r.e. set B so that $B \leq_{\mathbf{T}} A$ and so that the following requirements are satisfied for every $e \in N$:

$\mathbf{N}_e$ $\Psi_e(B) \neq A$.

$\mathbf{R}_e$ $\Phi_e(A) = U_e \Rightarrow U_e$ is recursive or there is nonrecursive $Q_e \leq_{\mathbf{T}} U_e, B$.

Here $\{\Psi_e\}_{e\in N}$ is an effective listing of all recursive functionals and $\{\Phi_e, U_e\}_{e\in N}$ is an effective listing of all pairs of functionals Φ and r.e. sets U. The Requirement $\mathbf{R}_e$ will be met by meeting the following requirements for every $i \in N$:

$$\mathbf{R}_{e,i} \quad \Phi_e(A) = U_e \Rightarrow U_e \text{ is recursive or } Q_e \leq_{\mathbf{T}} U_e, B \text{ and } Q_e \neq \overline{W}_i.$$

Besides meeting the requirements $\mathbf{R}_{e,i}$ and $\mathbf{N}_e$, we will ensure that $B \leq_{\mathbf{T}} A$ by permitting; i.e., if $x \in B_{s+1} - B_s$, we will have that $(\exists y \leq x)[y \in A_{s+1} - A_s]$. In this section, we describe the strategy for meeting a single requirement $\mathbf{R}_{e,i}$. Fix e, i. We will drop the subscripts in the ensuing description so that the requirement to be met may be restated as

$$\mathbf{R} \quad \Phi(A) = U \Rightarrow U \text{ is recursive or } Q \leq_{\mathbf{T}} U, B \text{ and } Q \neq \overline{W}.$$

Throughout the discussion, if g is a function that depends on s, $g(s)$ will denote the value of g at the beginning of stage s. If the value of g changes during the course of stage s, $g(s)$ denotes the current value of g at that point in the construction. For the sake of describing the action of $\mathbf{R}$, we define $l(s)$ and $\phi(x, s)$ to be the length of agreement and use functions in the computation $\Phi(A) = U$ at stage s. We will assume that U is enumerated in such a way so that if $x \in U_{s+1} - U_s$, then $\Phi_s(A_s; x) = 1$. This is permissible since we only care about those sets U for which $\Phi(A) = U$.

To meet $\mathbf{R}$ we will appoint **followers** such that at any stage s, at most two followers are **active** (uncancelled). The active followers at stage s are denoted $f1(s)$ and $f2(s)$. A follower x is **realized** at stage s if $x \in W_s$. If $f1(s)$ is defined then $f1(s)$ will be realized and if $f2(s)$ is defined then $f2(s)$ will not be realized. In addition, we will have that $f1(s) < f2(s)$ if both $f1(s)$ and $f2(s)$ are defined. Active followers may or may not be in **gaps** at any stage. Each follower x will have a **trace** $t(x, s)$ at each stage s such that x is active. The traces are for the purpose of showing that $Q \leq_{\mathbf{T}} B$. We will say that R is satisfied at stage s if there exists x such that $x \in W_s \cap Q_s$.

CONSTRUCTION. *Stage* $s+1$. $\mathbf{R}$ requires attention at stage $s+1$ if $W_s \cap Q_s = \emptyset$ and one of the following cases applies. If one does, take the action indicated.

Case a: There is an active follower x of $\mathbf{R}$ such that x is not in a gap, and $A_s[u(x, s)] \neq A_{s+1}[u(x, s)]$.

Action: Let x be the least such follower. Enumerate $t(x, s)$ into B. Declare that x and the active follower of $\mathbf{R}$ greater than x (if any) are in a gap.

Case b: Not case (a) and if x is the least active follower of $\mathbf{R}$ in a gap at stage s and that gap was opened at $s' + 1 < s$, then $l(s) > l(s')$.

Action: Close the gap of x and the larger active follower, if any. Adopt the subcase below that applies.

SUBCASE b1: x is realized and $U_s[x] \neq U_{s'}[x]$.

Action: Enumerate x into Q. Cancel any other followers of $\mathbf{R}$.

SUBCASE b2: Otherwise. (Either x is not realized or $U_s[x] = U_{s'}[x]$.) Redefine $t(y, s+1)$ for the active followers y of $\mathbf{R}$ so that $t(y, s+1) \geq u(y, s)$ and so that $t(y, s+1)$ is increasing in y.

Case c: Not cases (a) or (b) but $y = f2(s)$ is realized at stage s.

Action: Then set $f1(s+1) = y$, and set $f2(s+1)$ undefined. (This cancels the previous follower $f1(s)$.)

Case d: Not cases (a), (b), or (c), no follower of $\mathbf{R}$ is in a gap at s, $f2(s)$ is not defined, and there is an $x > l(s)$ such that $x > y$ for any previous follower y of $\mathbf{R}$.

Action: Let $x = f2(s+1)$. Define $t(x, s+1) \geq \max\{u(x, s), t(y, s) : y \leq x\}$. (Say $t(x, s+1) = s+1$.)

END OF CONSTRUCTION.

LEMMA 2.1: *$B \leq_{\mathrm{T}} A$. (In fact, $B \leq_{\mathrm{wtt}} A$.)*

Proof: A number y is enumerated into B at stage $s+1$ of the construction only if case (a) applies to y. In this case $y = t(x, s)$ for a follower x of $\mathbf{R}$ and $A_s[u(x, s)] \neq A_{s+1}[u(x, s)]$. If y was first appointed as a trace for x at stage $s' < s$, then $y = t(x, s') \geq u(x, s')$. Furthermore $u(x, s') = u(x, s)$, since s is the first stage after s' at which a gap is opened for x and hence for which $A_s[u(x, s)] \neq A_{s+1}[u(x, s)]$. Thus $y = t(x, s) \geq u(x, s)$ so that $B \leq_{\mathrm{T}} A$ by simple permitting. ∎

LEMMA 2.2: *Suppose that $\Phi(A) = U$. Then $Q \leq_{\mathrm{T}} U$. (In fact, $Q \leq_{\mathrm{wtt}} U$.)*

Proof: The hypothesis implies that every gap closes. Enumeration into Q happens only in subcase (b1). Thus if x is enumerated into Q at stage $s+1$, it is because a gap is closed for x at stage $s+1$ which was opened at some stage

$s' + 1$ for which $U_{s'}[x] \neq U_s[x]$. The reduction procedure for $Q \leq_T U$ is now clear. ∎

LEMMA 2.3: *Suppose that $\Phi(A) = U$. Then $Q \leq_T B$.*

Proof: The hypothesis that $\Phi(A) = U$ implies that for any follower x, only finitely many gaps can be opened for x and that each of these gaps closes. At the stage s that x is appointed as a follower, x receives a trace $y = t(x, s)$. This trace remains appointed to x unless x ceases to become active (in which case $x \notin Q$) or x enters a gap. Say x first enters a gap at stage $s' + 1$. Then $t(x, s') = t(x, s)$ is enumerated into B so that using B we can determine that such a stage exists. When the gap closes, say at $s'' + 1$, either x is enumerated into Q (and we answer accordingly) or x receives a new trace $t(x, s'' + 1)$. Again we can determine from $t(x, s'' + 1)$ and B whether x ever enters a gap after $s'' + 1$. We continue this process until either x enters Q or x has assigned a trace which never enters B. ∎

LEMMA 2.4: *If $\Phi(A) = U$ then U is recursive or $Q \neq \overline{W}$.*

Proof: If Subcase (b1) ever obtains, then $Q \neq \overline{W}$ and **R** never again receives attention. Thus suppose that subcase (b1) never obtains. Suppose that **R** has a follower y such that $y \notin W$. Then if y is appointed at stage $s + 1$, $y = f2(s + 1)$ and the values of $f1$ and $f2$ never change after stage $s+1$ since y is never realized. In this case y witnesses that $Q \neq W$ and **R** receives attention only finitely often since we can only open finitely many gaps for y. Suppose then that such a y does not exist. Then every follower of **R** is eventually realized and, by case (d), **R** has infinitely many followers. To recursively determine if $z \in U$, let x be a realized follower and s a stage such that $x \geq z$, x is realized at s, $l(s) > x$ and x is not in a gap at stage s. Then $U_s[x] = U[x]$. To see this, notice that for all stages $t \geq s$ there is a realized follower $y \geq x$ active at stage t. Let t be least such that $U_t[x] \neq U_{t+1}[x]$. By the convention on the enumeration of U, $l(t) < x$ so that at some stage s' such that $s < s' < t$, $A_{s'}[u(x, s')] \neq A_{s'+1}[u(x, s')]$ and a gap was opened for y by stage $s' + 1$ that is not closed by t. But when this gap is closed, subcase (b1) applies to y. ∎

Note that the possible outcomes of the module for **R** include the following:

(a) **R** is satisfied by virtue of $W \cap Q \neq \emptyset$. In this case **R** receives attention only finitely often.

(b) There is a single follower x of $\mathbf{R}$ such that x has infinitely many gaps. In this case $\Phi(A) \neq U$.

(c) $\lim_s f1(s) = \infty$. In this case, if $\Phi(A) = U$, then U is recursive.

(d) None of the above. Then only finitely many followers are appointed and each receives attention only finitely often so that $\Phi(A) \neq U$.

3. The priority tree

The construction to meet all the requirements $\mathbf{R}_{e,i}$ and $\mathbf{N}_e$ is a tree of strategies construction. We first construct the priority tree. The priority tree PT will have four different types of nodes.

The first type of node will be devoted to one of the requirements $\mathbf{R}_e$ and will be used to test the hypothesis $\Phi_e = U_e$. We call such a node an $\mathbf{R}_e$-node. The possible outcomes of this node will be denoted ∞ and f, which indicate that there are infinitely many or finitely many expansionary stages, respectively. A second type of node will be devoted to one of the requirements $\mathbf{N}_e$ and will be called an $\mathbf{N}_e$-node. The possible outcomes of an $\mathbf{N}_e$-node are natural numbers j with outcome j denoting that the length of agreement of the computations mentioned in requirement $\mathbf{N}_e$ is j. A third type of node is an $\mathbf{R}_{e,i}$-node. Such a node will be devoted to employing the basic module of Section 2 to meet $\mathbf{R}_{e,i}$. The outcomes of such a node are chosen from the set $\Lambda = \{f, g_3, g_2, g_1, w\}$. The outcomes correspond to the following dispositions of the basic module:

f	one of the finitary success outcomes,
g_3	$\lim_s f1(s) = \infty$,
g_2	infinitely many gaps for a single $f1$ follower,
g_1	infinitely many gaps for a single $f2$ follower,
w	one of the waiting outcomes.

Finally, we will have $\mathbf{T}$-nodes. Each $\mathbf{T}$-node will be an immediate predecessor of an $\mathbf{N}_e$-node σ and will be used to supply σ with a guess as to the recursive injury set to $\Phi_e(B) \neq A$ in a way to be described later. Each outcome of a $\mathbf{T}$-node β will be a finite set of nodes $\tau \subseteq \beta$ such that τ is an $\mathbf{R}_{f,j}$-node.

An ordering $\leq$ on nodes of the tree is defined lexicographically by using the following orderings at each level:

$\infty < f$	for outcomes of $\mathbf{R}_e$-nodes,
$f < g_3 < g_2 < g_1 < w$	for outcomes of $\mathbf{R}_{e,i}$-nodes,
$j < k$	for $k < j$ outcomes of $\mathbf{N}_e$-nodes,
$F < F'$	if $F \supseteq F'$ for outcomes of $\mathbf{T}$-nodes.

The reader should note that the order type of the outcomes of an $\mathbf{N}_e$-node is ω^*. The last of the above orderings does not determine a linear order; any linear order of the outcomes of a $\mathbf{T}$-node satisfying that property will suffice. We will use $\sigma <_L \tau$ to mean that $\sigma \leq \tau$ and $\sigma \not\subseteq \tau$. We think of the tree as growing down with the root at the top and outcomes at each level arranged from left to right in increasing order of $<$. Thus $\sigma <_L \tau$ if σ is to the left of τ in this tree.

We build the tree inductively. For each node σ on the tree we will associate to σ three lists, $L_1(\sigma)$, $L_2(\sigma)$ and $L_3(\sigma)$, and we will assign a requirement to σ. The lists will determine which requirement we assign to σ. List $L_1(\sigma)$ is the set of indices e of requirements $\mathbf{R}_e$ which still need attention on the path through σ, list $L_2(\sigma)$ is the set of indices e of requirements $\mathbf{N}_e$ which need such attention, and list $L_3(\sigma)$ is the set of pairs $\langle e,i\rangle$ such that $\mathbf{R}_{e,i}$ needs attention. For σ a node, let σ^- denote the immediate predecessor of σ on the tree. Also, let λ denote the empty sequence which is the root node of the tree. The next definition defines the lists; it depends on the definition of **injures** which we give later in this section.

Definition 3.1: Let $L_1(\lambda) = L_2(\lambda) = \omega$ and $L_3(\lambda) = \emptyset$. Now suppose that σ is a node for which L_1, L_2, and L_3 have not yet been defined. Then define the lists according to the following cases. (It will not be necessary to define the lists for σ a $\mathbf{T}$-node.)

CASE 1: σ^- is an $\mathbf{R}_e$-node. Then let $L_1(\sigma) = L_1(\sigma^-) - \{e\}$ and $L_2(\sigma) = L_2(\sigma^-)$. If $\sigma = \sigma^{-\frown} f$ then let $L_3(\sigma) = L_3(\sigma^-)$ but if $\sigma = \sigma^{-\frown}\infty$ then let $L_3(\sigma) = L_3(\sigma^-) \cup \{\langle e,i\rangle : i \in \omega\}$.

CASE 2: σ^- is an $\mathbf{N}_e$-node. Let $L_1(\sigma) = L_1(\sigma^-)$, $L_2(\sigma) = L_2(\sigma) - \{e\}$, and $L_3(\sigma) = L_3(\sigma^-)$.

CASE 3: σ^- is an $\mathbf{R}_{e,i}$-node. There are two subcases. If $\sigma = \sigma^{-\frown} w$ or $\sigma = \sigma^{-\frown} f$ then let $L_1(\sigma) = L_1(\sigma^-)$, $L_2(\sigma) = L_2(\sigma^-)$, and $L_3(\sigma) = L_3(\sigma^-) - \{\langle e,i\rangle\}$. If $\sigma = \sigma^\frown g_k$ for some k then $L_1(\sigma) = \{f > e : f \in \omega\}$, $L_2(\sigma) = \{f \geq e\colon f \in \omega\}$ and $L_3(\sigma) = L_3(\sigma^-) \cup \{\langle f,j\rangle\colon \sigma \text{ injures } \mathbf{R}_{f,j}\} - \{\langle f,j\rangle\colon f \geq e \text{ and } j \in \omega\}$.

Having defined the lists, we now assign a requirement to σ as follows. Let i be the least element of $L_1(\sigma) \cup L_2(\sigma) \cup L_3(\sigma)$. If $i \in L_1(\sigma)$, let σ be an $\mathbf{R}_i$-node. If $i \in L_2(\sigma) - L_1(\sigma)$ then let σ be a $\mathbf{T}$-node and let the successors of σ each be $\mathbf{N}_i$-nodes. Otherwise $i = \langle e, f\rangle \in L_3(\sigma)$ and we let σ be an $\mathbf{R}_{e,f}$-node.

The definition of the lists above and the assignment of requirements to nodes is the usual one for constructions of this type, except for the provision in case 3 which adds certain extra requirements to the path below $\sigma^{-\frown}g_k$. This addition is entailed by the new feature of our construction which we now describe. We need the following definition:

Definition 3.2: Suppose that σ is an $\mathbf{R}_{e,i}$-node. Let $\tau(\sigma)$ denote the longest initial segment of σ such that $\tau(\sigma)$ is an $\mathbf{R}_e$-node. ($\tau(\sigma)$ exists by the construction of the lists above. Requirement $\mathbf{R}_{e,i}$ is not assigned to a node unless requirement $\mathbf{R}_e$ has been assigned to some initial segment.)

In usual tree arguments, we would allow a node σ to act only at σ-stages, stages at which the guesses coded by σ look correct. However, for σ an $\mathbf{R}_{e,i}$-node, this restriction does not get along well with the basic module which requires us to open a gap at *any stage at which A permits.* We cannot assume that there are enough σ stages at which this happens. Thus we need to be able to open a gap at any stage. This contrasts with other "gap-cogap" tree constructions, where such gap opening was confined to σ stages. In our construction, we still will only be allowed to close the gap at $\tau(\sigma)$ stages. However, we will only be able to *appoint new followers* at σ stages. In the construction, this will be implemented by constructing a link from $\tau(\sigma)$ to σ at the stage at which we open a gap. The gap will be closed at a true $\tau(\sigma)$ stage by traveling the link. The consequence of the more frequent gap opening is that it is possible that nodes α such that $\tau(\sigma) \subset \alpha \subset \sigma$ do not *ever again* get a chance to act. That is, there are nodes on the "true path" which are not accessible infinitely often and so themselves are not "true." For such a node α, we need to place α's requirement on the tree again below σ. Note that the usual tree machinery automatically does this replacement if α's requirement has lower global priority than that of σ ($\mathbf{R}_i$, $\mathbf{N}_i$, or $\mathbf{R}_{i,j}$ for $i > e$). However, we must ensure that this is done for requirements of higher global priority that have lower local priority than $\tau(\sigma)$. That is the force of the following definition:

Definition 3.3: Suppose that $\sigma = \sigma^{-\frown}g_k$ for some k such that σ^- is an $\mathbf{R}_{e,i}$-

node. Then σ **injures** $\mathbf{R}_{f,j}$ if $f < e$ and there is β an $\mathbf{R}_{f,j}$-node such that $\tau(\sigma^-) \subseteq \beta \subseteq \sigma^-$.

This new property of the construction also requires the $\mathbf{T}$-nodes. The function of a $\mathbf{T}$-node σ is as follows. The node $\beta = \sigma\widehat{\ }F$ which follows it is an $\mathbf{N}_e$-node for some e. Thus β is attempting to establish that $\Phi_e(B) \neq A$. The strategy for this is the typical Sacks' strategy of preserving lengths of agreement. However, a $\mathbf{R}_{f,j}$-node τ for $\tau \subseteq \beta$ can injure this computation infinitely often by enumerating an infinite (recursive) sequence of traces into B. In a typical tree construction of this sort, this does not present a problem since the outcome of the $\mathbf{R}_{f,j}$-node τ provides information to β sufficient to determine whether the set of injuries is infinite or not. Namely, if β guesses that the outcome of τ is one of the gapping outcomes, then β guesses that τ contributes an infinite recursive set to B. With this in mind, β does not believe a computation $\Phi_{e,s}(B_s; x) = A_s(x)$ unless B_s is correct with respect to the hypothesized injury set. This suffices to make the injury set finite along the true path which is all that we need in order to meet $\mathbf{N}_e$. However, in our construction which follows, it is possible that β sees such an $\mathbf{R}_{f,j}$-node τ such that $\tau\widehat{\ }g_k \subseteq \beta$ but yet τ acts only finitely often (and the outcome τ is not correct) due to the strategy of gapping described above. Thus σ needs to know that τ will not contribute an infinite injury set to B. This is precisely what the $\mathbf{T}$-node is intended to accomplish. The guess F of σ is the guess as to which $\mathbf{R}_{f,j}$-nodes $\tau \subseteq \sigma$ will actually contribute an infinite set of traces to B. Then β uses that information to determine which computations to believe. It is clear that if F is the correct guess, then β will succeed as in typical constructions. We can now say precisely which sets F are such that $\sigma\widehat{\ }F$ should be a successor of a $\mathbf{T}$-node.

Definition 3.4: Suppose that σ is a $\mathbf{T}$-node. Then the immediate successors of σ are exactly the nodes $\sigma\widehat{\ }F$ for each F such that F is a subset of

$$\{\alpha\colon \alpha \text{ there are } e,\ i, \text{ and } k \text{ with } \alpha \text{ an } \mathbf{R}_{e,i}\text{-node and } \alpha\widehat{\ }g_k \subseteq \sigma\}.$$

The next lemmas guarantee that each infinite path γ of the priority tree PT has enough nodes so that all the requirements are satisfied. It will be helpful to have this picture of γ in mind. Along γ we place $\mathbf{R}_e$-nodes in decreasing order of priority. After we place an $\mathbf{R}_e$-node τ on the tree, we begin placing $\mathbf{R}_{e,i}$-nodes for each i. Each of the $\mathbf{R}_{e,i}$-nodes has an outcome on γ. If one of these nodes σ has a gapping outcome on γ (i.e. $\sigma \subset \sigma\widehat{\ }g_k \subset \gamma$) then this gives us a global win for requirement $\mathbf{R}_e$. If this happens, we do three things. We restart all requirements $\mathbf{R}_j$ for $j > e$ by placing new $\mathbf{R}_j$-nodes below σ. We also stop placing $\mathbf{R}_{e,k}$-nodes for any k. Finally, we start placing $\mathbf{R}_{j,k}$-nodes for requirements $\mathbf{R}_{j,k}$ which were injured by $\sigma^\frown g_k$. (Note that each of these requirements has $j < e$.) Thus below τ there will either be finitely many $\mathbf{R}_{e,i}$-nodes ending in one with a gapping outcome or for every i there will be an $\mathbf{R}_{e,i}$-node σ below τ, necessarily with outcome s or w, implying that each of the subrequirements $\mathbf{R}_{e,i}$ is met. The requirements $\mathbf{N}_e$ are placed in the tree at appropriate points corresponding to their priority relative to that of the requirements $\mathbf{R}_e$. We would omit the proof of these lemmas except that the added condition in Definition 3.1 of "injures" requires us to add $\mathbf{R}_{e,i}$-nodes to the tree more often than usual constructions. Indeed an $\mathbf{R}_{e,i}$-node σ may be added because of the behavior of a requirement $\mathbf{R}_f$ for $f > e$. However, in this case we will have $f \leq \langle e,i\rangle$ so that we can still show that this injury happens to requirement $\mathbf{R}_{e,i}$ only finitely often.

The first of the two lemmas establishes this last fact.

Lemma 3.5: *Suppose that $\sigma = \sigma^{-\frown} g_k$ is an $\mathbf{R}_{f,j}$ node and σ injures $\mathbf{R}_{e,i}$. Then $f \leq \langle e,i\rangle$.*

Proof: Otherwise, let σ be the least counterexample and let β be least such that σ injures $\mathbf{R}_{e,i}$ via β. In other words, σ^- is an $\mathbf{R}_{f,j}$-node for some $\langle f,j\rangle$, β is an $\mathbf{R}_{e,i}$-node, $f > \langle e,i\rangle$, and

$$\tau(\sigma^-) \subseteq \beta \subseteq \sigma^- \subset \sigma^{-\frown} g_k = \sigma.$$

Now $\langle e,i\rangle < f$ implies that since $\tau(\sigma^-)$ is not an $\mathbf{R}_{e,i}$-node, $\langle e,i\rangle \notin L_3(\tau(\sigma^-))$. Thus $\langle e,i\rangle \in L_3(\beta)$ implies that for some δ with $\tau(\sigma^-) \subset \delta \subseteq \beta$, $\langle e,i\rangle \in L_3(\delta) - L_3(\delta^-)$. Thus either $\mathbf{R}_{e,i}$ is injured by δ or δ is a gapping outcome of an $\mathbf{R}_{l,m}$-node for $l < e$. In the latter case $l \leq \langle e,i\rangle$ by the properties of the pairing function and in the former case $l \leq \langle e,i\rangle$ by the hypothesis that σ is the least counterexample to this fact. Thus $l < f$ and so $f \in L_1(\delta)$. But then there must be an $\mathbf{R}_f$-node τ' such that $\delta \subseteq \tau' \subseteq \sigma^-$. This contradicts the fact that $\tau(\sigma^-) \subset \delta$. ∎

Lemma 3.6: *Suppose that γ is an infinite path through the priority tree. Then for every $e \in \omega$,*

(a) *there are only finitely many $\mathbf{R}_e$-nodes τ such that $\tau \subset \gamma$,*

(b) *there are only finitely many nodes* σ *such that* σ *is an* $\mathbf{R}_{e,i}$*-node for some* i *and* $\sigma^\frown g_k \subset \gamma$,

(c) *for every* i, *there are only finitely many* $\mathbf{R}_{e,i}$*-nodes* σ *such that* $\sigma \subset \gamma$.

Proof: We first prove (a) and (b) by simultaneous induction on e. Suppose then that (a) and (b) are true for all $i < e$. By Definition 3.1, if $e \notin L_1(\sigma^-)$, then $e \notin L_1(\sigma)$ unless σ^- is an $\mathbf{R}_{i,j}$ node for some $i < e$ and $\sigma = \sigma^{-\frown} g_k$ for some k. By (b) and the inductive hypothesis there are only finitely many such σ on γ. For each such σ there is at most one $\mathbf{R}_e$-node on γ. This proves (a).

By (a) of the inductive hypothesis, let $\tau \subset \gamma$ be such that for $i \leq e$ there are no $\mathbf{R}_i$-nodes τ' extending τ on γ. By (b) we may also assume that for all $i < e$ there are no $\mathbf{R}_{i,j}$-nodes σ extending τ on γ with gapping outcomes on γ. Let σ be such that σ is an $\mathbf{R}_{e,i}$-node for some i and $\tau \subseteq \sigma^\frown g_k \subset \gamma$. By Definition 3.1, $\langle e, j\rangle \notin L_3(\sigma^\frown g_k)$ for any j. (It appears that we have met $\mathbf{R}_e$ globally at $\sigma^\frown g_k$.) Suppose for a contradiction of (b) that j and α are such that $\sigma \subset \alpha$ and $\langle e, j\rangle \in L_3(\alpha) - L_3(\alpha^-)$. Then it must be the case that case (3) of Definition 3.1 applies to α^- and either α injures $\mathbf{R}_{e,j}$ or α is an $\mathbf{R}_{f,k}$-node with a gapping outcome on γ and $f < e$. The latter does not happen by the inductive hypothesis. If the former is true then we have that $\tau(\alpha^-) \subset \sigma \subset \sigma^\frown g_k \subset \alpha^-$. But this implies that $f \in L_1(\sigma^\frown g_k)$ and thus that there is an $\mathbf{R}_f$-node τ' such that $\sigma^\frown g_k \subseteq \tau' \subseteq \alpha^-$. But then $\tau(\alpha^-) \supseteq \tau'$ contrary to the hypothesis that $\tau(\alpha^-) \subset \sigma$. (At the gapping outcome of $\mathbf{R}_{e,i}$, we restart requirement $\mathbf{R}_j$ and must first put an $\mathbf{R}_j$-node on the tree before any $\mathbf{R}_{j,k}$-node.) Thus (b) is proved.

To prove (c), fix γ, e and i. Let τ be the longest $\mathbf{R}_e$-node on γ (if there are no $\mathbf{R}_e$-nodes on γ, then (c) is trivially true, otherwise τ exists by (a)). By the proof of (b), if there is any j and $\mathbf{R}_{e,j}$-node σ with a gapping outcome on γ, then there are no $\mathbf{R}_{e,i}$-nodes on γ extending σ so (c) is proved in this case. Otherwise suppose that σ and σ' are two $\mathbf{R}_{e,i}$-nodes on γ such that $\tau \subset \sigma \subset \sigma'$. Then it must be the case that $\langle e, i\rangle \in L_3(\sigma')$ so that there is β such that $\sigma \subset \beta \subseteq \sigma'$ and $\langle e, i\rangle \in L_3(\beta) - L_3(\beta')$. This can only be because $\mathbf{R}_{e,i}$ is injured by β. But β injures $\mathbf{R}_{e,i}$ if β^- is an $\mathbf{R}_{f,k}$-node for some $f \leq \langle e, i\rangle$ (by the previous lemma) and β has a gapping outcome on γ. By (b) there are only finitely many such β. Hence there are only finitely many such pairs σ and σ' and so only finitely many $\mathbf{R}_{e,i}$-nodes on γ. ∎

In the construction that follows, Q_τ denotes the version of Q being built at $\mathbf{R}_e$-

node τ. To initialize a node means to cancel all followers and traces associated with the node.

4. The construction

CONSTRUCTION. STAGE $s+1$: Stage $s+1$ has two steps each of which is executed in turn.

STEP 1: Suppose that α is an $\mathbf{R}_{e,i}$-node and there is an x such that x is an active follower of α at $s+1$, $t(x,s)$ is defined, $A_{s+1}[u(x,\alpha,s)] \neq A_s[u(x,\alpha,s)]$, and there is no link $(\tau(\alpha),\alpha)$ defined at stage s.

Action: Enumerate $t(x,s)$ into B, construct a link $(\tau(\alpha),\alpha)$, initialize all nodes γ such that $\alpha <_L \gamma$, and initialize all α' such there is a link $(\tau(\alpha),\alpha')$ such that $\tau(\alpha) \subset \alpha \subset \alpha'$.

STEP 2: We define TP_{s+1}, the apparent true path at stage $s+1$ in substages below. At substage t, we define an initial segment $\mathrm{TP}(t,s+1)$ of TP_{s+1}. If $\sigma = \mathrm{TP}(t,s+1)$ for some substage t of stage $s+1$, we say that $s+1$ is a **genuine** σ-stage. If $\sigma \subseteq \mathrm{TP}_{s+1}$ we say that $s+1$ is a σ-stage. If $s+1$ is a σ-stage we will initialize all nodes γ with $\sigma <_L \gamma$.

SUBSTAGE 0: Define $\mathrm{TP}(0,s+1) = \lambda$. (Note that λ is the only $\mathbf{R}_0$ node in the priority tree PT.) See if $s+1$ is λ-expansionary. That is, see if $l(\lambda,s) > \max\{l(\lambda,s'),0,n\colon s' < s\}$, where $l(\lambda,s)$ denotes the length of agreement between $\Phi_0(A)$ and U_0 at stage s, and n denotes the largest number associated with any node ν with $\tau(\nu) = \lambda$. If $s+1$ is not λ-expansionary, then set $\mathrm{TP}(1,s+1) = f$ and go to substage 1. If $s+1$ is λ-expansionary, then adopt the first case below which pertains and then go to substage 1.

CASE 1: There is a link (λ,ν) for some node ν.

Action: Set $\mathrm{TP}(1,s+1) = \nu$.

CASE 2: Otherwise.

Action: Set $\mathrm{TP}(1,s+1) = \infty$.

SUBSTAGE $t+1$: If $|\mathrm{TP}(t,s+1)| = s$, then let $\mathrm{TP}_{s+1} = \mathrm{TP}(t,s+1)$ and proceed to the next stage. Otherwise, let $\sigma = \mathrm{TP}(t,s+1)$ and adopt the first case below which pertains.

CASE 1: σ is a $\mathbf{R}_{e,i}$-node.

SUBCASE 1.1: $W_{\sigma,s} \cap Q_{\tau(\sigma),s} \neq \emptyset$.

Action: Let $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }f$. Initialize σ if $s+1$ is the first $\sigma\hat{\ }f$-stage.

SUBCASE 1.2: $W_{\sigma,s} \cap Q_{\tau(\sigma),s} = \emptyset$, and there is a link $(\tau(\sigma), \sigma)$. (This means that we have just traveled the link.) Cancel the link and adopt the first subcase below that applies.

SUBCASE 1.2a: There is a realized follower $x = f1(\sigma, s)$ such that $U_{\tau(\sigma),s}[x] \neq U_{\tau(\sigma),s'}[x]$ where s' denotes the largest genuine $\tau(\sigma)$-stage $< s$.

Action: Enumerate x into $Q_{\tau(\sigma),s+1}$. Let $\mathrm{TP}_{s+1} = \sigma\hat{\ }f$.

SUBCASE 1.2b: Subcase 1.2a does not pertain and the σ-gap is now an $f1(\sigma, s)$-gap. (That is, $A_{s+1}[u(f1(\sigma, s), \sigma, s')] \neq A_{s'}[u(f1(\sigma, s), \sigma, s')]$ where s' denotes the largest genuine $\tau(\sigma)$-stage $< s$.)

Action: Set $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }g_2$. Redefine $t(y, s+1)$ for the active followers y of σ so that $t(y, s+1)$ is fresh and exceeds $s+1$. (And hence exceeds all uses at this stage.)

SUBCASE 1.2c: As in subcase 1.2b except that the σ-gap is $f2(\sigma, s)$-gap. (Thus $A_{s+1}[u(f2(\sigma, s), \sigma, s')] \neq A_{s'}[u(f2(\sigma, s), \sigma, s')]$ but $A_{s+1}[u(f1(\sigma, s), \sigma, s')] = A_{s'}[u(f1(\sigma, s), \sigma, s')]$ where s' denotes the largest genuine $\tau(\sigma)$-stage $< s$.)

Action: Set $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }g_1$. Redefine $t(f2(\sigma, s+1), s+1) = t(f2(\sigma, s), s+1)$ to be large and fresh.

SUBCASE 1.2d: We are not in a σ-gap at all. (This will mean that the link is there for the sake of defining a τ-use for $f2(\sigma, s)$.)

Action: Define $t(f2(\sigma, s), s+1)$ to be large and fresh. Set $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }w$.

SUBCASE 1.3: Neither subcase 1.2 nor subcase 1.1 pertains but $s+1$ is a genuine $\tau(\sigma)$ stage.

SUBCASE 1.3a: $f2(\sigma, s) \in W_{\sigma,s}$.

Action: Define $f1(\sigma, s+1) = f2(\sigma, s)$ and declare that $f2(\sigma, s+1) \uparrow$. Set $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }g_3$.

SUBCASE 1.3b: $f2(\sigma, s)\uparrow$.

Action: Define $f2(\sigma, s+1)$ to be large and fresh. Construct a link $(\tau(\sigma), \sigma)$. Set $\mathrm{TP}_{s+1} = \sigma$ and proceed to the next stage.

SUBCASE 1.3c: $f2(\sigma, s)\downarrow$ but $f2(\sigma, s) \notin W_{\sigma,s}$.

Action: Set $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }w$.

SUBCASE 1.4: None of Subcases 1.1, 1.2 or 1.3 hold.

Action: Set $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }w$.

CASE 2: $\sigma = \mathrm{TP}(t, s+1)$ is a $\mathbf{T}$-node.

Action: For each outcome F of σ, let $s(F)$ denote the maximum of 0 and the most recent genuine $\sigma\hat{\ }F$-stage, if any. Find the leftmost outcome F which has appeared correct since $s(F)$. That is, for each $\nu \in F$, since stage $s(F)$ there has been a genuine ν-stage. (Note that this method of determining F implies that if there are infinitely many $\sigma\hat{\ }F_1$ stages and $\sigma\hat{\ }F_2$ stages then there are infinitely many $\sigma\hat{\ }F$ stages for some $F \supseteq F_1 \cup F_2$.) Define $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }F$.

CASE 3: $\sigma = \mathrm{TP}(t, s+1)$ is a $\mathbf{N}_e$-node.

Action: This will be a Sacks-type requirement implemented on a tree. Note that the predecessor ν of σ is a $\mathbf{T}$-node and $\sigma = \nu\hat{\ }F$ for some set F of $\mathbf{R}_{e,i}$-nodes γ such that $\gamma \subseteq \sigma$. (F represents a guess as to which $\mathbf{R}_{e,i}$-nodes are visited infinitely often and so can injure the computation at σ infinitely often.) Compute the σ-correct length of agreement between $\Psi_{\sigma,s}(B_s; x)$ and A_s as follows. We will say that a computation $\Psi_{\sigma,s}(B_s; x)\downarrow$ is σ-correct if the use $u = u(\Psi_{\sigma,s}(B_s; x))$ of the computation satisfies the following. For all $\gamma \in F$, if $\gamma\hat{\ }g_2 \subseteq \sigma$ or $\gamma\hat{\ }g_3 \subseteq \sigma$ then $t(f1(\gamma, s), s) > u$, and if $\gamma\hat{\ }g_1 \subseteq \sigma$ we have $t(f2(\gamma, s), s) > u$. (In other words, a computation looks correct to σ if it has been cleared of all traces that σ believes will eventually enter B based on the guesses given by the intial segments of σ.) Let i be the σ-correct length of agreement and let $\mathrm{TP}(t+1, s+1) = i+1$.

CASE 4: $\sigma = \mathrm{TP}(t, s+1)$ is a $\mathbf{R}_e$-node. See if $s+1$ is σ-expansionary. If $s+1$ is not σ-expansionary at substage t, then let $\mathrm{TP}(t+1, s+1) = \sigma\hat{\ }f$. If $s+1$ is σ-expansionary at substage t, then adopt the first subcase below to pertain.

SUBCASE 4.1: There is a link (σ, ν) for some node ν.

Action: Let $\mathrm{TP}(t+1, s+1) = \nu$.

SUBCASE 4.2: Otherwise.

Action: Set $\mathrm{TP}(t+1, s+1) = \sigma^\frown\infty$.

END OF CONSTRUCTION.

5. The verification

We define TP the true path to be the leftmost path visited infinitely often. (As usual, we define this inductively via: $\lambda \subset \mathrm{TP}$ and for all ν, if $\nu \subset \mathrm{TP}$, then for r an outcome of ν, $\nu^\frown r \subset \mathrm{TP}$ iff there are infinitely many $\nu^\frown r$-stages and, for all $r' <_L r$, there are only finitely many $\nu^\frown r'$-stages.) We define the *genuine* true path GTP to be those $\sigma \subset \mathrm{TP}$ such that additionally there are infinitely many *genuine* σ-stages. Now for this construction, it is not at all clear that GTP is infinite. In fact, it is not obvious that TP is infinite because of the ω^* ordering of the outcomes at an $\mathbf{N}_e$-node. The crucial lemma is the following:

LEMMA 5.1 (Golden Path Lemma): GTP *is infinite. Furthermore, if σ is on* GTP *and is a* $\mathbf{N}_e$*-node, then* $\mathbf{N}_e$ *is satisfied.*

Proof: Obviously, λ is on GTP. Now suppose that σ is on GTP. We need to show that for some $\gamma \supseteq \sigma$, γ is on GTP.

Because σ is on GTP, there is a stage s_0 such that if $s \geq s_0$ is a γ-stage and $\gamma \leq_L \sigma$, then $\gamma \subseteq \sigma$. Now after stage s_0, no node β left of σ without a follower at stage s_0 will **ever** get a follower, since followers are appointed at genuine β-stages. Hence there are at most a fixed finite set of nodes left of σ which can ever be at the ends of links. Suppose that β is left of σ. Then after stage s_0 if there is a link to β, it is either permanent but never traveled, or it is cancelled and the node β is initialized. (If a link is cancelled then the bottom node is initialized.) Therefore we may also assume that for all $s \geq s_0$ and all $\beta <_L \sigma$ there are no links with bottom β created at stage s. Also, there may be a finite set of nodes $\eta \subset \sigma$ such that η is on TP but not GTP. We will assume that such η are not visited after stage s_0. If a node β is such that $\tau(\beta) = \eta$ for such an η, no follower can be appointed for β after s_0. (Followers for β can only be appointed at genuine η stages.) Therefore we can assume also that after stage s_0 there are no links constructed with tops $\eta \subseteq \sigma$ not on GTP.

The argument now divides into four cases, according to the type of the node σ. Suppose first that σ is a $\mathbf{T}$-node. Then whenever s is a genuine σ-stage, s is a genuine $\sigma\hat{\ }F$-stage for some F. Since there are only finitely many outcomes F of σ, for some F, $\sigma\hat{\ }F$ is on GTP.

Suppose next that σ is a $\mathbf{R}_{e,i}$-node. Since there are only finitely many outcomes of σ, we need only argue that at infinitely many genuine σ-stages, we play some outcome of σ. The only situation in which we do not play an outcome of σ at a genuine σ-stage, is when we create a link (τ, σ) in Subcase 1.3b of the construction (when $f2(\sigma, s)$ is undefined). This link remains in place until either it is cancelled, or we find a genuine τ-expansionary stage with the length above $f2(\sigma, s)$. In the latter case, we will play $\sigma\hat{\ }w$ at the relevant genuine τ-expansionary stage. In the former case suppose that we cancel the link (τ, σ) before we get to traverse it. Now by choice of s_0, the link can only be cancelled (i) by some node $\alpha <_L \sigma$ or (ii) by some $\alpha \subset \sigma$ at the same time as constructing a link (τ, σ) (where of course this is the same $\tau = \tau(\sigma)$). (The point is that only the creation of new links can cancel the link (τ, σ).) Case (i) is impossible since we have assumed that there will be no links with bottoms $\beta <_L \sigma$ created after stage s_0. And $\alpha <_L \sigma$. Therefore case (ii) holds. Notice that the link (τ, α) remains in place for the same reason until either it is traversed or replaced by some (τ, α') with $\alpha' \subset \alpha$. It follows that for some $\eta \subset \sigma$ we will eventually traverse (τ, η) and play an outcome of η. Now by choice of stage s_0, we cannot play an outcome $\eta\hat{\ }u <_L \sigma$. Therefore, since $\tau(\sigma) = \tau(\eta)$ it can only be that $u \in \{f, w\}$. (See Definition 3.1.) However, after we traverse a link such as (τ, η), we can only play an outcome from $\{f, g_1, g_2, g_3\}$. It follows that we play the outcome f of η and therefore η never again receives attention. In particular there is no link with bottom η ever created after the stage we traverse (τ, η). Therefore the next time we create a link (τ, σ) that is cancelled by the creation of a link (τ, α) for some $\alpha \subset \sigma$, we know that $\alpha \neq \eta$. There are only finitely many possible such nodes α and hence from some stage on, every link of the form (τ, σ) created will never be destroyed, and will be eventually traversed. But when the link (τ, σ) is traversed, an outcome of σ is played. Since we play outcomes of σ infinitely often, and σ has only finitely many outcomes to choose from, some outcome of σ is on GTP.

Suppose then that σ is a $\mathbf{N}_e$-node, devoted to the requirement that $\Psi_e(B) \neq A$. Each time we have a genuine σ-stage, then we must have a genuine $\sigma\hat{\ }j$-stage for some outcome $j \in \omega$. We will argue that there is a greatest number j such that outcome j is visited infinitely often. Thus suppose not. Fix $j \in \omega$ and let $s = s(j) > s_0$ and $j' > j$ be such that s is the first genuine $\sigma\hat{\ }j'$-stage. Such a j' exists by our hypothesis. By the cancellation process (and the ω^* priority ordering of the outcomes of σ) it follows that there are no nodes below σ with any followers or traces at stage $s(j)$. Now the predecessor γ of σ is a $\mathbf{T}$-node and is also on GTP. Therefore $\sigma = \gamma\hat{\ }F$ such that the information in F is correct. That is, for all $\mathbf{R}_{e,i}$-nodes $\nu\hat{\ }g_j \subset \sigma$, if ν is on GTP, then $\nu \in F$. In particular, the σ-correct computations at stage $s = s(j)$ will never be injured by any trace which is already appointed by stage s. Furthermore, all traces appointed after stage s cannot injure these computations. Therefore, it follows that the $\Psi_e(B_{s(j)}; j'')$-computations are correct for all $j'' \leq j'$. Now a' la' Sacks, we see that A is recursive. This argument shows both that some outcome $\sigma\hat{\ }j$ must be on GTP and also that $\mathbf{N}_e$ is met at σ.

To complete the proof of the lemma, we need consider the case that σ is a $\mathbf{R}_e$-node. Now as σ has only two outcomes, one must be on TP. If $\sigma\hat{\ }f$ is on TP then it is on GTP. So suppose that $\sigma\hat{\ }\infty$ is on TP, but not GTP. Then after some stage $s_1 > s_0$, at each genuine σ-stage we either play outcome f or we have a $\sigma\hat{\ }\infty$-stage that is not a genuine $\sigma\hat{\ }\infty$-stage.

Let $s_2 > s_1$ be such a $\sigma\hat{\ }\infty$-stage. It can only be that there is a link (σ, ν) traversed at stage s_2. To argue that GTP exists we need argue that in these circumstances, there is some ν on TP to which σ links infinitely often. The point is this. If σ links to a node ν at a stage $s_3 > s_1$, then there is no node η with $\tau(\eta) = \sigma$, and such that η gets a (new) follower at stage s_3. Followers to nodes η with $\tau(\eta) = \sigma$ are appointed only at *genuine* $\sigma\hat{\ }\infty$-stages. Since σ can only link to nodes ν with followers and such that $\tau(\nu) = \sigma$, it follows that after stage s_1, σ can only link to one of a finite number of nodes. Therefore σ will link to some highest priority ν infinitely often. Since ν's follower is never cancelled, it can only be that ν is on GTP. Thus σ has an extension on GTP. This completes the proof of the lemma. ∎

LEMMA 5.2 (Golden Rule Lemma): *For every* $e \in \omega$

(a) *there is an* $\mathbf{R}_e$*-node* β *on* GTP

(b) *there is an* $\mathbf{N}_e$*-node* β *on* GTP

(c) *if* τ *be the longest* $\mathbf{R}_e$*-node on* GTP *(which exists by* (a) *and Lemma* 3.6*), and if* $\tau\hat{\ }\infty$ *is on* TP *either*

  (i) *there exists* $i \in \omega$ *and* $\sigma \supset \tau$ *an* $\mathbf{R}_{e,i}$*-node on* GTP *such that* $\sigma\hat{\ }g_k$ *is on* GTP *for some* k *or*

(ii) *for every $i \in \omega$ there is $\mathbf{R}_{e,i}$-node $\sigma \supset \tau$ on* GTP.

Proof: To prove (a) and (b), let β be the longest $\mathbf{R}_e$-node ($\mathbf{N}_e$-node) on TP which exists by Lemma 3.6. We claim that β is on GTP. Suppose otherwise. Then it must be the case that there are τ and σ on GTP such that $\tau \subset \beta \subset \sigma$ and the link (τ, σ) is traversed at infinitely many τ stages. This implies that $\sigma^\frown g_k$ is on GTP for some k. Now τ is an $\mathbf{R}_f$-node for some f and σ an $\mathbf{R}_{f,j}$-node for some k. Now $f < e$ ($f \leq e$) by the way that requirements are assigned to the paths of the tree. (Since β is an $\mathbf{R}_e$-node or $\mathbf{N}_e$-node, an $\mathbf{R}_f$-node must be placed on the tree after β before any $\mathbf{R}_{f,j}$-node contradicting the claim that $\tau = \tau(\sigma) \subset \beta$.) But then by the construction of the tree, there are $\mathbf{R}_e$-nodes ($\mathbf{N}_e$-nodes) on TP above $\sigma^\frown g_k$ contradicting the hypothesis that β is the longest such.

To prove (c), let τ be the last $\mathbf{R}_e$-node on TP which by the proof above is also on GTP. Suppose first that there is an $i \in \omega$ and σ an $\mathbf{R}_{e,i}$-node such that $\sigma^\frown g_k$ is on TP for some k. We claim that $\sigma^\frown g_k$ is on GTP. Suppose not. Then as in the proof of (a), there is τ' and σ' on GTP which are $\mathbf{R}_f$- and $\mathbf{R}_{f,j}$-nodes respectively such that $\tau' \subset \sigma^\frown g_k \subset \sigma'$. Again, we must have $f < e$ because $\tau \subset \sigma$, but as in (a) and (b) this implies that τ is not the last $\mathbf{R}_e$-node on TP. Supposing there is no such i, it is clear by the definition of list L_3 that for every i there is an $\mathbf{R}_{e,i}$-node $\sigma \supset \tau$ on TP. Given i, let $\sigma \supset \tau$ be the longest $\mathbf{R}_{e,i}$-node extending σ on TP. We argue that σ is in fact on the GTP. Again, suppose not. As before, there are τ' and σ' on GTP which are $\mathbf{R}_f$- and $\mathbf{R}_{f,j}$-nodes respectively such that $\tau' \subset \sigma \subset \sigma'$ and $\sigma'^\frown g_k$ is on GTP. Since τ is the last $\mathbf{R}_e$-node on TP, it must be the case that $f > e$. But then by the definition of injures, we have that $\sigma'^\frown g_k$ injures $\mathbf{R}_{e,i}$ and so by the construction of the lists, σ is not the last $\mathbf{R}_{e,i}$-node on TP. This contradicts the assumption so that σ is in fact on GTP. ∎

LEMMA 5.3: *Suppose that $\tau^\frown\infty$ is on* GTP *and that τ is the last $\mathbf{R}_e$-node on* GTP. *Then $Q_\tau \leq_T U_\tau$.*

Proof: Since there are infinitely many genuine τ stages, each τ gap, opened via a link (τ, σ), is either closed or cancelled by the subsequent τ-expansionary stage. Thus, if we find the maximal τ-expansionary stage s where $U_s[x] = U[x]$, it must be that $x \in Q_\tau$ iff $x \in U_{\tau,s+1}$. ∎

LEMMA 5.4: *Under the hypotheses of Lemma 5.3, $Q_\tau \leq_T B$.*

Proof: Use the same proof as Lemma 2.3, modified as in Lemma 5.3. ∎

LEMMA 5.5 (Truth of Outcome Lemma): *If $\Phi_e(A) = U_e$, then U_e is recursive or $Q_\tau \neq W_i$ for all i, where τ is the longest node devoted to $\mathbf{R}_e$ on* GTP.

Proof: Let $\sigma \supseteq \tau^\frown\infty$ have $\mathbf{R}_{e,i}$ devoted to σ and suppose that σ is on GTP. Now every gap that is opened from τ to σ is eventually traversed or cancelled. Cancellation only happens if we are left of σ, and this can only happen finitely often as σ is on GTP. Thus, after some stage s_1, every link (τ, σ) is eventually traversed. Clearly if $\sigma^\frown w$ is on GTP, then since σ is only initialized finitely often, it can only be that some follower of σ is never realized. Hence $W_i \neq Q_\tau$ in that case as with the case that $\sigma^\frown f$ is on GTP. So suppose that $\sigma^\frown g_j$ is on GTP. Then the argument is the same as for Lemma 2.4. Each time $u(f2(\sigma, s), s)$ is permitted by A we open a gap. This gap is closed at the next genuine $\tau^\frown\infty$-stage. Since U does not change in the gap (by the outcome), we will reset $t(f2(\sigma, s), s)$, ready for the next gap. Therefore we can use the process of Lemma 2.3 to recursively compute U. ∎

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